

Embeddings of non-simply-connected 4-manifolds in 7-space

II. On the smooth classification

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Abstract

We work in the smooth category. Let N be a closed connected orientable 4-manifold with torsion free H_1 , where $H_q := H_q(N; \mathbb{Z})$. Our main result is *a readily calculable classification of embeddings $N \rightarrow \mathbb{R}^7$ up to isotopy*, with an indeterminacy. Such a classification was only known before for $H_1 = 0$ by our earlier work from 2008. Our classification is complete when $H_2 = 0$ or when the signature of N is divisible neither by 64 nor by 9.

The group of knots $S^4 \rightarrow \mathbb{R}^7$ acts on the set of embeddings $N \rightarrow \mathbb{R}^7$ up to isotopy by embedded connected sum. In Part I we classified the quotient of this action. The main novelty of this paper is the description of this action for $H_1 \neq 0$, with an indeterminacy.

Besides the invariants of Part I, the classification involves a refinement of the Kreck invariant from our work of 2008 which detects the action of knots.

For $N = S^1 \times S^3$ we give a geometrically defined 1–1 correspondence between the set of isotopy classes of embeddings and a quotient of the set $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_{12}$.

1 Introduction

1.1 Overview and main results

Motivation and background for this paper may be found in Part I [CSI, §1]. We adopt the notation and setting of Part I [CSI, §1.1]. In particular we consider *smooth* manifolds, embeddings and isotopies and recall that:

- N is a closed connected orientable 4-manifold;
- $E^m(N)$ denotes the set of isotopy classes of embeddings $f: N \rightarrow S^m$.

In this paper we classify $E^7(N)$ when $H_1(N; \mathbb{Z})$ is torsion free (up to an indeterminacy in certain cases). See Theorems 1.1, 1.4 and Corollaries 1.3, 1.6 below. Our classification is complete when $H_2 = 0$ (see Theorem 1.4 and Corollary 1.6.b) or when the signature of N is divisible neither by 64 nor by 9 (see Theorem 1.4 and Corollary 1.3). The classification requires finding a complete set of invariants and constructing embeddings realizing particular values of these invariants. The invariants we use are described in [CSI, Lemma 1.3, §2.2, §2.3] and §2.1. An overview of the proof of their completeness is given in [CSI, §1.4] and in Remark 1.10 below.

The group $E^7(S^4) \cong \mathbb{Z}_{12}$ acts on $E^7(N)$ by embedded connected sum. This action was investigated in [Sk10] and determined when $H_1(N; \mathbb{Z}) = 0$ in [CS11], which also classified $E^7(N)$ in this

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case. In [CSI] we described the quotient $E_{\#}^7(N) := E^7(N)/E^7(S^4)$ when $H_1(N; \mathbb{Z}) = 0$. Thus the main novelty of this paper is the description of this action for $H_1(N; \mathbb{Z}) \neq 0$. Cf. Remark 1.9.

Denote by $q_{\#} : E^7(N) \rightarrow E_{\#}^7(N)$ the quotient map.

In order to state our main result for $N = S^1 \times S^3$ consider the following diagram (where the left triangle is not commutative):

$$\begin{array}{ccc} \mathbb{Z}_{12} \times \mathbb{Z}^2 & \xrightarrow{\text{pr}_2} & \mathbb{Z}^2 \\ \# \times \tau \downarrow & \searrow \tau & \downarrow \tau_{\#} := q_{\#} \tau \\ E^7(S^1 \times S^3) & \xrightarrow{q_{\#}} & E_{\#}^7(S^1 \times S^3). \end{array}$$

The maps $\tau, q_{\#}$ are defined in [CSI, §1.2]. We define

$$\# \times \tau \quad \text{by} \quad (\# \times \tau)(a, l, b) := a \# \tau(l, b) \quad \text{and} \quad \tau_{\#} := q_{\#} \tau.$$

Theorem 1.1. *The map $\# \times \tau : \mathbb{Z}_{12} \times \mathbb{Z}^2 \rightarrow E^7(S^1 \times S^3)$ is a surjection such that*

(a) *for different l, b the sets $P_{l,b} := (\# \times \tau)(\mathbb{Z}_{12} \times (l, b))$ either are disjoint or coincide;*

$$(b) \quad P_{l,b} = P_{l',b'} \quad \Leftrightarrow \quad (l = l' \quad \text{and} \quad b \equiv b' \pmod{2l});$$

$$(c) \quad |P_{l,b}| = \begin{cases} 12 & l \neq 0 \\ 2 \gcd(b, 6) & l = 0. \end{cases}$$

In Theorem 1.1 the surjectivity of τ and (a) and (b) follow from [CSI, Theorem 1.1]. The new part of Theorem 1.1 is (c); this part follows from Corollary 1.6.b below (because for $l \neq 0$ the group $\text{coker } \bar{l}$ is finite, so $\text{div } b = 0$). Cf. Addendum 2.8.

Example 1.2. There is an embedding $f : S^1 \times S^3 \rightarrow S^7$ with $f(N) \subset S^6$ and a pair of non-isotopic embeddings $g_1, g_2 : S^4 \rightarrow S^7$ such that $f \# g_1$ and $f \# g_2$ are isotopic.

This example follows because there is a representative of $\tau(0, 1)$ whose image is in $S^6 \subset S^7$ [CSI, Lemma 2.21] and $|P_{0,1}| = 2$ by Theorem 1.1.

Example 1.2 shows necessity of the assumption of simple-connectivity the following result (which is [Sk10, The Effectiveness Theorem 1.2]):¹

If $f : N \rightarrow S^7$ is an embedding of a spin simply-connected closed 4-manifold N , $f(N) \subset S^6$ and embeddings $g_1, g_2 : S^4 \rightarrow S^7$ are not isotopic, then $f \# g_1$ and $f \# g_2$ are not isotopic.

Before stating our main result in the general case we state the following corollary of it.

Corollary 1.3 (of Theorem 1.4.c). (a) *If $\kappa(f)$ is neither divisible by 4 nor by 3, then for each embedding $g : S^4 \rightarrow S^7$ the embeddings $f \# g$ and f are isotopic.*

(b) *If $\kappa(f)$ is divisible by 4 but neither by 8 nor by 3, then there is a non-trivial embedding $g_1 : S^4 \rightarrow S^7$ such that for every embedding $g : S^4 \rightarrow S^7$ the embedding $f \# g$ is isotopic to either f or $f \# g_1$.*

Corollary 1.3 follows from Corollary 1.6.bc or from Theorem 1.4.c (because $4\mathbb{Z}_{\gcd(\kappa(f), 24)} = 0$ under the assumptions of Corollary 1.3). The assumption of Corollary 1.3.a is automatically satisfied when the signature of N is divisible neither by 16 nor by 9. Cf. Remark 1.8.a.

¹It follows that the Effectiveness Theorem 1.2 of the earlier versions of [Sk10] was false.

For stating our main result in the general case we use the conventions, notation and definitions of [CSI, §1.2]. Denote

$$\widehat{n} := \gcd(n, 24).$$

If $H_1 = 0$, then the map $\varkappa_{\#}$ (which is the Boéchat-Haefliger invariant defined in [CSI, §1.2, §2.2]) is 1-1 and $\varkappa = \varkappa_{\#} q_{\#}$ is surjective:

$$E^7(N) \xrightarrow{q_{\#}} E^7_{\#}(N) \xrightarrow{\varkappa_{\#}} H_2^{DIFF} := \{u \in H_2 \mid \rho_2 u = w_2^*(N), u \cap_N u = \sigma(N)\} \subset H_2.$$

For each $u \in H_2^{DIFF}$ we have $|\varkappa^{-1}(u)| = \widehat{u} / \gcd(u, 2)$.²

Our second main result is a generalization of the above statement to non-simply-connected 4-manifolds. The maps \varkappa , λ , $\beta_{u,l}$, and $\eta_{u,l,b}$, $\theta_{u,l,b}$ of Theorem 1.4 below are defined in [CSI, §2.2, §2.3] and in §2.1.

Definition of \cap_d . For a symmetric pair (u, l) and $d := \operatorname{div} u \in \mathbb{Z}$ a bilinear map

$$\cap_d: \operatorname{coker}(2\rho_d \bar{l}) \times \ker(2\rho_d \bar{l}) \rightarrow \mathbb{Z}_d \quad \text{is well-defined by} \quad [c] \cap_d y := c \cap_N y.^3$$

Theorem 1.4. *Let N be a closed connected orientable 4-manifold with torsion free H_1 .*

(a) *The product*

$$\varkappa \times \lambda: E^7(N) \rightarrow H_2^{DIFF} \times B(H_3)$$

has non-empty image consisting of symmetric pairs.

(b) *For each $(u, l) \in \operatorname{im}(\varkappa \times \lambda)$ denote $d := \operatorname{div} u$. Each map*

$$\beta_{u,l}: (\varkappa \times \lambda)^{-1}(u, l) \rightarrow \operatorname{coker}(2\rho_d \bar{l})$$

is surjective (see the remark immediately below the Theorem).

(c) *For each $b \in \operatorname{coker}(2\rho_d \bar{l})$ the map*

$$\theta_{u,l,b}: \ker(2\rho_d \bar{l}) \rightarrow 4\mathbb{Z}_{\widehat{d}}$$

is a homomorphism and each map

$$\eta_{u,l,b}: \beta_{u,l}^{-1}(b) \rightarrow \frac{\mathbb{Z}_{\widehat{d}}}{\operatorname{im} \theta_{u,l,b}}$$

is an injection whose image consists of all even elements (see the remark immediately below the Theorem). Moreover,

$$\theta_{u,l,b}(y) - \theta_{u,l,b'}(y) = 4\rho_{\widehat{d}}(b - b') \cap_d y \quad \text{for each } y \in \ker(2\rho_d \bar{l}) \subset H_3.$$

$$(d) \quad |\beta_{u,l}^{-1}(b)| = \frac{\widehat{u}}{\gcd(u, 2) \cdot |\operatorname{im} \theta_{u,l,b}|}.$$

²This is proven in [Sk10, CS11] building on [BH70].

³Indeed, for each $x \in H_3$ and $y \in \ker(2\rho_d \bar{l})$ we have $2\bar{l}x \cap_N y = 2l(x, y) \equiv_d 2l(y, x) = 2\bar{l}y \cap_N x \equiv_d 0$. Hence $\operatorname{im}(2\rho_d \bar{l}) \cap_N \ker(2\rho_d \bar{l}) = \{0\} \subset \mathbb{Z}_d$.

Remark on relative invariants. We call geometrically defined maps invariants (this is informal, formally an invariant is the same as a map). The maps λ and \varkappa are invariants. The maps $\beta_{u,l}$ and $\eta_{u,l,b}$ are *relative invariants*. For $\eta_{u,l,b}$ this means that for $[f_0], [f_1] \in \beta_{u,l}^{-1}(b)$ there is an invariant $([f_0], [f_1]) \mapsto \eta(f_0, f_1)$ (defined in §2.1) and that $\eta_{u,l,b}(f) := \eta(f, f')$ for a fixed choice of $[f'] \in \beta_{u,l}^{-1}(b)$. We suppress the choice of $[f']$ from the notation. For $\beta_{u,l}$ the situation is similar.

Parts (a) and (b) of Theorem 1.4 follow from [CSI, Theorem 1.2]. The new part of Theorem 1.4 is (c), which is proven in §2.1. Part (d) follows because by (c) $\text{im } \eta_{u,l,b} = 2\mathbb{Z}_{\widehat{d}} / \text{im } \theta_{u,l,b}$.

We remark that Theorem 1.1 is not an immediate corollary of Theorem 1.4, cf. [CSI, Remarks 1.4.a and 2.16].

Identify $E^7(S^4)$ and \mathbb{Z}_{12} by any isomorphism.

Addendum 1.5. In the notation of Theorem 1.4, for each $a \in \mathbb{Z}_{12}$ and $f \in \beta_{u,l}^{-1}(b)$

$$\eta_{u,l,b}(f \# a) = \eta_{u,l,b}(f) + [2a] \in \frac{\mathbb{Z}_{\widehat{d}}}{\text{im } \theta_{u,l,b}}.$$

This follows from the definition of $\eta_{u,l,b}$ (§2.1) and [CSI, Lemma 4.3.b].

Corollary 1.6. For each $(u, l) \in \text{im}(\varkappa \times \lambda)$ let $d := \text{div } u$. There is $f_{u,l} \in (\varkappa \times \lambda)^{-1}(u, l)$ such that for each $f \in (\varkappa \times \lambda)^{-1}(u, l)$ and $a, a' \in \mathbb{Z}_{12}$, denoting $b := \beta(f, f_{u,l}) \in \text{coker}(2\rho_d \bar{l})$ we have

- (a) $f \# a = f \# a' \Leftrightarrow a = a'$, provided
 - $u = 0$ and $\text{div } b$ is divisible by 6, or
 - $u \neq 0$, $2\rho_d \bar{l} = 0$ and u is divisible by $24 \text{ord}(4b)$;
- (b) $f \# a = f \# a' \Leftrightarrow a \equiv a' \pmod{2 \text{gcd}(\text{div } b, 6)}$, provided $u = 0$;
- (c) $f \# a = f \# a' \Leftrightarrow a \equiv a' \pmod{\frac{\widehat{u}}{\text{ord}(4b) \text{gcd}(u, 2)}}$, provided $u \neq 0$ and $2\rho_d \bar{l} = 0$.

The class u is divisible by d and hence by the order $\text{ord}(4b)$ of d in the d -group $\text{coker}(2\rho_d \bar{l})$.

Part (a) follows from Parts (b,c). Parts (b,c) are proven in §2.2. Cf. Remark 2.6 below.

Corollary 1.7. Theorem 1.4 has the following restatement analogous to Theorem 1.1 and to [CSI, Corollary 2.14.b]. There is a surjection

$$\tau : \mathbb{Z}_{12} \times H_1 \times H_2^{DIFF} \times B_0(H_3) \rightarrow E^7(N) \quad \text{such that}$$

$$\tau(a, b, u, l) = \tau(a', b', u', l') \Leftrightarrow u = u', \quad l = l', \quad b - b' \in \ker(2\rho_{\text{div } u} \bar{l}_u) \quad \text{and} \quad a - a' \in \text{im } \eta_{u, l_u, b},$$

where $l_u := l + \lambda\tau(0, 0, u, 0)$.

1.2 Discussion of the action of knots

Remark 1.8 (The action of knots in Theorem 1.4). (a) Take any $[f] \in E^7(N)$. Let

$$O(f) = O([f]) := \{[f \# g] : [g] \in E^7(S^4)\}$$

be the orbit of $[f]$ under the action of $E^7(S^4)$. We have $O(f) = \beta_{u,l}^{-1}(b)$ when $[f] \in \beta_{u,l}^{-1}(b)$ by [CSI, Theorem 1.2] and the additivity of \varkappa, λ and β [CSI, Lemmas 2.3 and 2.9].

Define the *inertia group* of f , $I(f) \subset E^7(S^4) = \mathbb{Z}_{12}$,⁴ to be the subgroup of isotopy classes in $E^7(S^4)$ which do not change $[f]$ after embedded connected sum:

$$I(f) = I([f]) := \{[g] \in E^7(S^4) : [f \# g] = [f]\}$$

For some cases this orbit and group are found in terms of u, l, b in Corollaries 1.3 and 1.6.

(b) Problem: characterize those f for which $|O(f)| = 12$ (i.e. $|I(f)| = 1$), and those f for which $|O(f)| = 1$ (i.e. $|I(f)| = 12$).

(c) The indeterminacy in the classification of Theorem 1.4.c corresponds to the fact that we do not always know $\text{im } \theta_{u,l,b}$. Thus determining $\text{im } \theta_{u,l,b}$ becomes a key problem. This image is found in this paper when either $u = 0$ or $2\rho_d \bar{l} = 0$ (Corollary 1.6) or in the cases (1,2,3) of Remark 2.6 below. For general u, l (and even simple enough N) there are some b for which the methods of this paper do not completely determine $\text{im } \theta_{u,l,b}$.

Remark 1.9 (The action of knots in general). If the quotient $E_{\#}^m(P)$ is known for a closed n -manifold P , the description of $E^m(P)$ is reduced to the *determination of the orbits* of the embedded connected sum action of $E^m(S^n)$ on $E^m(P)$. For a general closed n -manifold P describing the action by a non-zero group of knots $E^m(S^n)$ on $E^m(P)$ is a non-trivial task. For the cases when the quotient $E_{\#}^7(N)$ coincides with the set of PL embeddings up to PL isotopy, the quotient has been known since 1960s [MAE, MAF, MAT]. However, until recently no description of the action (or, equivalently, no classification of $E^m(P)$) was known for $E^m(S^n) \neq 0$ and P not a disjoint union of homology spheres. For recent results see [Sk08'] and [Sk10, CS11] mentioned above. On the other hand, the description of the action in [CRS07, Sk11, CRS12, Sk15] is not hard, the hard part of the cases considered there is rather the description of the quotient $E_{\#}^7(N)$.

There are non-isotopic embeddings $g_1, g_2: S^2 \rightarrow S^4$ and an embedding $f: \mathbb{R}P^2 \rightarrow S^4$ such that $[f \# g_1] \neq [f \# g_2]$ [Vi73]. I.e. the action of the monoid $E^4(S^2)$ on $E^4(\mathbb{R}P^2)$ is not free.

Various authors have studied the analogous connected sum action of the group of homotopy n -spheres on the set of smooth n -manifolds homeomorphic to given manifold; see for example [Sc73, Wi74] and references there.

Remark 1.10 (An approach to the action of knots). Let us explain the ideas required to move from the classification modulo knots in [CSI] to the main results of this paper. We briefly recall and continue the discussion in [CSI, 1.4].

Suppose that $f_0, f_1: N \rightarrow S^7$ are embeddings. Assume that f_1 is isotopic to $f_0 \# g$ for some embedding $g: S^4 \rightarrow S^7$. By [CSI, Isotopy Classification Modulo Knots Theorem 2.8] this is equivalent to $\lambda(f_0) = \lambda(f_1)$, $\varkappa(f_0) = \varkappa(f_1)$ and $\beta(f_0, f_1) = 0$. The complements C_0 and C_1 may be glued together along a bundle isomorphism $\varphi: \partial C_0 \rightarrow \partial C_1$ to form a parallelizable closed 7-manifold $M = C_0 \cup_{\varphi} (-C_1)$. Recall that $d := \text{div } \varkappa(f_0)$ is the divisibility of $\varkappa(f_0) \in H_2$. By the assumption on f_0, f_1 there is a joint Seifert class $Y \in H_5(M)$ such that $\rho_d Y^2 = 0$, i.e. a d -class [CSI, Lemma 4.1]. There is a spin null-bordism (W, z) of (M_{φ}, Y) , since $\Omega_7^{\text{Spin}}(\mathbb{C}P^{\infty}) = 0$. Since $\rho_d Y^2 = 0$, the class $\rho_d z^2 \in H_4(W, \partial; \mathbb{Z}_d)$ lifts to $\bar{z}^2 \in H_4(W; \mathbb{Z}_d)$. Recall that $p_W^* \in H_4(W, \partial)$ is the Poincaré dual of p_W , the spin Pontrjagin class of W . We then verified that *the Kreck invariant*,

$$\eta(\varphi, Y) := \bar{z}^2 \cap_W \rho_{\hat{d}}(z^2 - p_W^*) \in \mathbb{Z}_{\hat{d}},$$

⁴The inertia group of f is just the stabilizer of $[f]$ under the action of $E^7(S^4)$. We use the word ‘inertia’ following its use for the action of the group homotopy spheres on the diffeomorphism classes of smooth manifolds: see the second paragraph of Remark 1.9.

determines the surgery obstruction for W to be spin diffeomorphic to the product $C_0 \times I$ [CSI]. We proved that $\eta(\varphi, Y)$ is independent of the choices of W, z, \bar{z}^2 for a fixed bundle isomorphism φ and d -class Y [CSI, §4.1]. We also proved that $\eta(\varphi, Y)$ is independent of the choice of φ : for the precise statement, see [CSI, Lemma 4.3.c]. So we need to know the various values of $\eta(\varphi, Y)$ arising from the different possible choices of Y . These choices are described in [CSI, Description of d -classes for M_f Lemma 4.7]. The achievement of this paper is showing that the change of $\eta(\varphi, Y)$ under a change of Y is precisely determined by $\theta_{u,l,b}$, and proving the properties of $\theta_{u,l,b}$ (Lemma 2.1).

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2 Definition of invariants and proofs modulo Lemma 2.1

2.1 Definition of the η - and θ -invariants and proof of Theorem 1.4.c

In this paper we use the notation and definitions of [CSI, §§2.1-2.3, 4.1]. In particular, we assume that groups $H_q := H_q(N; \mathbb{Z})$ are torsion free. Denote $\theta(f, y) := \eta(\text{id } \partial C_f, Y_{f,y}) \in \mathbb{Z}_{\widehat{d}}$.

Lemma 2.1 (proved in §3.2, §3.3). *(a) $\theta(f, y)$ is divisible by 4 for each f, y .*

(b) The map $\theta(f, \cdot) : \ker(2\rho_d \lambda(f)) \rightarrow \mathbb{Z}_{\widehat{d}}$ is a homomorphism, where $d := \text{div } \varkappa(f)$.

(c) For each $(u, l) \in \text{im}(\varkappa \times \lambda)$, $d := \text{div } u$, representatives f_0, f_1 of two isotopy classes in $(\varkappa \times \lambda)^{-1}(u, l)$ and $y \in \ker(2\rho_d \bar{l})$ we have $\theta(f_0, y) - \theta(f_1, y) = 4\rho_{\widehat{d}}(\beta(f_0, f_1) \cap_N y)$.

Definition of $\theta_{u,l,b}$. Take any $(u, l) \in \text{im}(\varkappa \times \lambda)$ and $b \in K_{u,l}$. Let $d := \text{div } u$. Define

$$\theta_{u,l,b} : \ker(2\rho_d \bar{l}) \rightarrow 4\mathbb{Z}_{\widehat{d}} \quad \text{by} \quad \theta_{u,l,b}(y) := \theta(f, y), \quad \text{where} \quad [f] \in \beta_{u,l}^{-1}(b).$$

The map $\theta_{u,l,b}$ is well-defined (i.e. is independent of the choice of f) and is a homomorphism by Lemma 2.1.ab and the transitivity of β [CSI, Lemma 2.10].

Definition of $\eta(f_0, f_1)$. Take representatives f_0, f_1 of two isotopy classes in $(\varkappa \times \lambda)^{-1}(u, l)$ such that $\beta(f_0, f_1) = 0$. By [CSI, Lemma 2.5] there is a π -isomorphism $\varphi : \partial C_0 \rightarrow \partial C_1$. By [CSI, Lemma 4.1] there is a d -class $Y \in H_5(M_\varphi)$. Define

$$\eta(f_0, f_1) := [\eta(\varphi, Y)] \in \frac{\mathbb{Z}_{\widehat{d}}}{\text{im } \theta_{u,l,b}}.$$

This is well-defined by [CSI, Lemma 4.3.c] and Lemma 2.3.a below, and is even by [CSI, Lemma 4.3.a].

Lemma 2.2. *Let $f_0, f_1, f_2 : N \rightarrow S^7$ be embeddings, $\varphi_{01} : \partial C_0 \rightarrow \partial C_1$ and $\varphi_{12} : \partial C_1 \rightarrow \partial C_2$ π -isomorphisms, $Y_{01} \in H_5(M_{\varphi_{01}})$ and $Y_{12} \in H_5(M_{\varphi_{12}})$ d -classes. Then $\varphi_{02} := \varphi_{12}\varphi_{01}$ is a π -isomorphism and there is a d -class $Y_{02} \in H_5(M_{\varphi_{02}})$ such that $\eta(\varphi_{02}, Y_{02}) = \eta(\varphi_{01}, Y_{01}) + \eta(\varphi_{12}, Y_{12})$.*

This is proved analogously to [CS11, Lemma 2.10], cf. [Sk08', §2, Additivity Lemma] (the property that Y_{02} is a d -class is achieved analogously to [CSI, proof of Lemma 4.6]).

Lemma 2.3. *Take representatives f_0, f_1 of two isotopy classes in $(\varkappa \times \lambda)^{-1}(u, l)$ such that $\beta(f_0, f_1) = 0$. Denote $d := \text{div } u$.*

(a) *For each π -isomorphism $\varphi : \partial C_0 \rightarrow \partial C_1$ the residue $\eta(f_0, f_1)$ is independent of the choice of a d -class $Y \in H_5(M_\varphi)$.*

(b) *If $\eta(f_0, f_1) = 0$, then for each π -isomorphism $\varphi : \partial C_0 \rightarrow \partial C_1$ there is a d -class $Y \in H_5(M_\varphi)$ such that $\eta(\varphi, Y) = 0 \in \mathbb{Z}_{\widehat{d}}$.*

Proof of (a). Take any d -classes $Y', Y'' \in H_5(M_\varphi)$. Part (a) follows because

$$\eta(\varphi, Y') - \eta(\varphi, Y'') \stackrel{(1)}{=} \eta(\text{id } \partial C_0, Y) \stackrel{(2)}{=} \theta(f_0, y) = \theta_{u, l, \beta(f_0, f')}(y) \in \mathbb{Z}_{\widehat{d}},$$

where

- equality (1) holds for some d -class $Y \in H_5(M_{f_0})$ by Lemma 2.2;
- equality (2) holds for some $y \in \ker(2\rho_d \bar{l})$ by description of d -classes [CSI, Lemma 4.7]. \square

Proof of (b). Take any π -isomorphism $\varphi : \partial C_0 \rightarrow \partial C_1$. Part (b) follows because

$$0 \stackrel{(1)}{=} \eta(\varphi, Y') - \theta_{u, l, \beta_{u, l}(f_0)}(y) = \eta(\varphi, Y') - \theta(f_0, y) \stackrel{(3)}{=} \eta(\varphi, Y) \in \mathbb{Z}_{\widehat{d}},$$

where

- equality (1) holds for some d -class $Y' \in H_5(M_\varphi)$ and $y \in \ker(2\rho_d \bar{l})$ because $\eta(f_0, f_1) = 0$;
- equality (3) holds for some d -class $Y \in H_5(M_\varphi)$ by Lemma 2.2. \square

Lemma 2.4 (Transitivity of η). *For each three embeddings $f_0, f_1, f_2 : N \rightarrow S^7$ having the same values of \varkappa - and λ -invariants and such that $\beta(f_0, f_1) = \beta(f_1, f_2) = 0$ we have $\eta(f_2, f_0) = \eta(f_2, f_1) + \eta(f_1, f_0)$.*

This follows by Lemma 2.2.

Theorem 2.5 (Isotopy classification). *If $\lambda(f_0) = \lambda(f_1)$, $\varkappa(f_0) = \varkappa(f_1)$, $\beta(f_0, f_1) = 0$ and $\eta(f_0, f_1) = 0$, then f_0 is isotopic to f_1 .*

Proof. Analogously to the proof of [CSI, Isotopy Classification Modulo Knots Theorem 2.8]. Only replace the second paragraph by ‘Since $\eta(f_0, f_1) = 0$, by Lemma 2.3.b we can change Y and assume additionally that $\eta(\varphi, Y) = 0$.’ \square

Definition of $\eta_{u, l, b}$. Take any $[f_0] \in \beta_{u, l}^{-1}(b)$. Define a map

$$\eta_{u, l, b} : \beta_{u, l}^{-1}(b) \rightarrow \frac{\mathbb{Z}_{\widehat{d}}}{\text{im } \theta_{u, l, b}} \quad \text{by} \quad \eta_{u, l, b}[f] := \eta(f, f_0).$$

The map $\eta_{u, l, b}$ depends on f_0 but we do not indicate this in the notation.

Proof of Theorem 1.4.c. The property on $\theta_{u, l, b} - \theta_{u, l, b'}$ holds by Lemma 2.1.c. The map $\eta_{u, l, b}$ is injective by the Isotopy Classification Theorem 2.5. The image of this map consists of all even elements by [CSI, Lemma 4.3.a] and Addendum 1.5. \square

2.2 Proof of Corollary 1.6.bc

Remark 2.6. In Corollary 1.6.bc the assumption ‘ $d = 0$ or $2\rho_d\bar{l} = 0$ ’ can be replaced by each of the following successively weaker assumptions

- (1) $\rho_{\hat{d}}\ker(2\rho_d\bar{l}) \subset \rho_{\hat{d}}H_3$ is a direct summand, or
- (2) each homomorphism $\rho_{\hat{d}}\ker(2\rho_d\bar{l}) \rightarrow 4\mathbb{Z}_{\hat{d}}$ extends to $\rho_{\hat{d}}H_3$, or
- (3) there is an element $\tilde{b} \in \text{coker}(2\rho_d\bar{l})$ such that $\theta_{u,l,\tilde{b}} = 0$.

Clearly, ‘either $d = 0$ or $2\rho_d\bar{l} = 0$ ’ \Rightarrow (1) \Rightarrow (2).

Denote $X_{u,l} := \ker(2\rho_{\text{div}(u)}\bar{l}) \subset H_3$.

Proof of (2) \Rightarrow (3). Take any $b' \in \text{coker}(2\rho_d\bar{l})$. We have $\theta_{u,l,b'} = \theta_{u,l,b'}^+ \rho_{\hat{d}}$ for some homomorphism $\theta_{u,l,b'}^+ : \rho_{\hat{d}}X_{u,l} \rightarrow 4\mathbb{Z}_{\hat{d}}$. Extend $\theta_{u,l,b'}^+$ to a homomorphism $\rho_{\hat{d}}H_3 \rightarrow 4\mathbb{Z}_{\hat{d}}$. Since H_3 is free, $\rho_{\hat{d}}H_3$ is a free $\mathbb{Z}_{\hat{d}}$ -module. Hence the latter homomorphism is divisible by 4. Then by Poincaré duality there is a class $x \in \rho_{\hat{d}}H_1$ such that $\theta_{u,l,b'}^+(z) = 4x \cap_N z$ for each $z \in \rho_{\hat{d}}X_{u,l}$. Let $\tilde{b} := b' + [\tilde{x}]$, where $\tilde{x} \in \rho_dH_1$ is a lifting of x . Then by Theorem 1.4

$$\theta_{u,l,\tilde{b}}(y) = \theta_{u,l,b'}(y) - 4\rho_{\hat{d}}([\tilde{x}] \cap_N y) = \theta_{u,l,b'}^+(\rho_{\hat{d}}y) - 4x \cap_N \rho_{\hat{d}}y = 0 \quad \text{for each } y \in X_{u,l}.$$

□

Proof of the formula of Corollary 1.6.bc under the assumption (3) of Remark 2.6. Define $\beta'_{u,l}(f) := \tilde{b} - \beta_{u,l}(f)$. Then $\theta'_{u,l,b} = \theta_{u,l,\tilde{b}-b}$ for each $b \in \text{coker}(2\rho_d\bar{l})$, hence $\theta'_{u,l,0} = 0$. Therefore we may assume that $\beta_{u,l}$ is chosen so that $\theta_{u,l,0} = 0$.

Take any $b \in \text{coker}(2\rho_d\bar{l})$ and denote $X_{b,u,l} := 4b \cap_d X_{u,l}$. So

$$\gcd(d, 2) \cdot |\beta_{u,l}^{-1}(b)| \stackrel{(1)}{=} \frac{\hat{d}}{|\text{im } \theta_{u,l,b}|} \stackrel{(2)}{=} \frac{\hat{d}}{|\rho_{\hat{d}}X_{b,u,l}|} = [\mathbb{Z}_{\hat{d}} : \rho_{\hat{d}}X_{b,u,l}] \stackrel{(4)}{=} \gcd(\hat{d}, [\mathbb{Z}_d : X_{b,u,l}]) = [\widehat{\mathbb{Z}_d : X_{b,u,l}}],$$

where equalities (1) and (2) hold by Theorem 1.4. Now the formula of Corollary 1.6.bc is implied by the following Lemma 2.7. □

Lemma 2.7. *Let V be a free \mathbb{Z} -module, d an integer and $m : V \rightarrow V^*$ a homomorphism whose polarization $V \times V \rightarrow \mathbb{Z}$ has a symmetric mod d reduction. Then for each $c \in \text{coker}(\rho_d m)$*

$$[\mathbb{Z}_d : c(\ker(\rho_d m))] = \begin{cases} \text{div } c & d = 0 \\ \frac{d}{\text{ord } c} & d \neq 0 \end{cases}$$

($c(\ker(\rho_d m)) \subset \mathbb{Z}_d$ is defined analogously to definition of \cap_d before Theorem 1.4).

Proof for $d = 0$. We need to prove the following.

Let V be a free \mathbb{Z} -module and $m : V \rightarrow V^$ a homomorphism whose polarization $V \times V \rightarrow \mathbb{Z}$ is symmetric. Then for each $c \in V^*$ we have $[\mathbb{Z} : c(\ker m)] = \text{div } c$.*

Let us prove the ‘ \subset ’ part. Assume that q is a divisor of $c + \text{Tors coker } m$. Then there exist $s, l_1, \dots, l_s \in \mathbb{Z}$ and $c_0, t_1, \dots, t_s \in V^*$ such that $l_n t_n \in \text{im } m$ for each $n = 1, \dots, s$ and $c = qc_0 + t_1 + \dots + t_s$. Take any $y \in \ker m$. Since the polarization of m is symmetric, we have $l_n t_n(y) = 0 \in \mathbb{Z}$, thus $t_n(y) = 0$. Hence $c(y) = qc_0(y) + (t_1 + \dots + t_s)(y) = qc_0(y)$ is divisible by q .

Let us prove the ‘ \supset ’ part. Assume that q is a divisor of $c|_{\ker m}$. The subgroup $\ker m \subset V$ is a direct summand. Take a decomposition $V = \ker m \oplus V'$. Since $m|_{V'} : V' \rightarrow \operatorname{im} m$ is an isomorphism, there is an element $x \in V'$ such that $c|_{V'} = m(x)|_{V'}$. Since the polarization of m is symmetric, $m(x)|_{\ker m} = 0$. Then $c - m(x)$ coincides with c on $\ker m$ and is zero on V' . So $c - m(x) = qc_0$ for some $c_0 \in V^*$. Hence $d(c + \operatorname{Tors} \operatorname{coker} m)$ is divisible by q . \square

Proof for $d \neq 0$. We need to prove that $|c(\ker(\rho_d m))| = \operatorname{ord} c$ for each $c \in \operatorname{coker}(\rho_d m)$. (We remark that this is obvious for $\rho_d m = 0$, which case is sufficient for Corollary 1.6.c.)

Denote $K := \rho_d \ker(\rho_d m) \subset V/dV$. Since the polarization $V \times V \rightarrow \mathbb{Z}$ of m has a symmetric mod d reduction, $\operatorname{im}(\rho_d m) \subset K^\perp \subset V^*/dV^*$. Since $|\operatorname{im}(\rho_d m)| = \frac{|V/dV|}{|K|} = |K^\perp|$, it follows that $\operatorname{im}(\rho_d m) = K^\perp$. Now the required assertion follows because for each $c' \in V^*/dV^*$

$$|c'(K)| = \frac{d}{\operatorname{div} c'(K)} = \min\{r \mid rc'(K) = 0\} = \min\{r \mid rc' \in K^\perp\} = \operatorname{ord}_{(V^*/dV^*)/K^\perp}(c' + K^\perp).$$

\square

Addendum 2.8. For each $l \in \mathbb{Z} - \{0\}$ there is a map $\psi_l : \mathbb{Z} \times \mathbb{Z}_{2l} \rightarrow \mathbb{Z}_{12}$ such that for each $a, a' \in \mathbb{Z}_{12}$ and $l, b, l', b' \in \mathbb{Z}$ we have $a \# \tau(l, b) = a' \# \tau(l', b')$ if and only if

$$\left[\begin{array}{l} \text{either } l = l' = 0, \quad b = b' \quad \text{and} \quad a \equiv a' \pmod{2 \operatorname{gcd}(b, 6)} \\ \text{or } l = l' \neq 0, \quad b \equiv b' \pmod{2l} \quad \text{and} \quad \rho_{12}(a - a') = \psi_l([b/2l], \rho_{2l}b) - \psi_l([b'/2l], \rho_{2l}b) \end{array} \right.$$

Proof. By Theorem 1.1.b if either $l \neq l'$ or $b \not\equiv b' \pmod{2l}$, then the equivalence is clear because none of the two assertions holds.

Assume that $l = l'$ and $b \equiv b' \pmod{2l}$. Let $\tau := a \# \tau(l, b)$ and $\tau' = a' \# \tau(l, b')$. By the Isotopy Classification Theorem 2.5 and Theorem 1.1.b $\tau = \tau' \Leftrightarrow \eta(\tau, \tau') = 0$. Since $\operatorname{div}(\varkappa(\tau)) = 0$, we may use Corollary 1.6.b.

If $l = 0$, then $b = b'$. By Theorem 1.1.b and Corollary 1.6.b $\operatorname{im} \theta_{0,0,b}$ is formed by elements of \mathbb{Z}_{24} divisible by $4 \operatorname{gcd}(b, 6)$. Hence by Addendum 1.5 and the transitivity of η (Lemma 2.4) $\eta(\tau, \tau') = \rho_{4 \operatorname{gcd}(b, 6)}(2a - 2a') \in \mathbb{Z}_{4 \operatorname{gcd}(b, 6)}$.

If $l \neq 0$, then by Theorem 1.1.b and Corollary 1.6.b $\operatorname{im} \theta_{0,l,b} = 0$ and $\eta(\tau, \tau') \in 2\mathbb{Z}_{24}$. For each $x \in \{0, 1, \dots, 2l - 1\}$ and $k \in \mathbb{Z}$ define $\psi_l(k, \rho_{2l}x) := \frac{1}{2}\eta(\tau(l, x), \tau(l, x + 2kl)) \in \mathbb{Z}_{12}$. Then by Addendum 1.5 and the transitivity of η (Lemma 2.4)

$$\eta(\tau, \tau') = \rho_{24}(2a - 2a') - \eta(\tau(l, \bar{b}), \tau(l, b)) + \eta(\tau(l, \bar{b}), \tau(l, b')) = \rho_{24}(2a - 2a') - \psi_l([\frac{b}{2l}], \rho_{2l}b) + \psi_l([\frac{b'}{2l}], \rho_{2l}b),$$

where $\bar{b} \in \{0, 1, 2, \dots, 2l - 1\}$ is uniquely defined by $b \equiv \bar{b} \pmod{2l}$.

These two formulas for $\eta(\tau, \tau')$ imply the equivalence describing preimages of τ . \square

3 Proof of Lemma 2.1 on θ -invariant

In this section we use homological Alexander duality and the restriction homomorphism defined in [CSI, §3.1].

3.1 Idea of proof of Lemma 2.1

We start with a lemma allowing to simplify the proof of the main result for $N = S^1 \times S^3$ (Theorem 1.1). The simplified proof is not presented, so the lemma is not used in the sequel. The standard embedding $\tau_0 : S^1 \times S^3 \rightarrow S^7$ is defined in [CSI, §2.4].

Lemma 3.1. $\eta(\tau_0, y) = 0$ for each $y \in H_3(S^1 \times S^3)$.

Proof. Define an extension

$$i : D^2 \times D^4 \rightarrow S^7 \quad \text{of } \tau_0 \text{ by } i(x, y) := (y\sqrt{2 - |x|^2}, 0, 0, x)/\sqrt{2}.$$

Take an embedding $v_0 : S^5 \rightarrow S^7 - i(S^1 \times D^4)$ whose linking coefficient with $i(S^1 \times D^0)$ is $y \cap_{S^1 \times S^3} [S^1 \times 1_3]$. We omit subscript τ_0 in this proof. Since $\text{lk}(\widehat{A}y, \tau_0(S^1 \times 1_3)) = y \cap_{S^1 \times S^3} [S^1 \times 1_3]$, we have $\widehat{A}y = [v_0(S^5)] \in H_5(C) \cong \mathbb{Z}$. We also have $\widehat{A}y = i_C \nu^! y$. Take a representative P of y and a chain

$$V \in C_6(C) \quad \text{such that} \quad \partial V = \nu^{-1}P - v_0(S^5).$$

Since C is parallelizable, v_0 extends to an orientation-preserving embedding $v_2 : S^5 \times D^2 \rightarrow \text{Int } C = \text{Int } C \times \frac{1}{2}$ transversal to V and such that $\text{im } v_2 \cap V = v_0(S^5)$. Extend v_2 to an orientation-preserving embedding $v_3 : S^5 \times D^3 \rightarrow \text{Int}(C \times I)$. Let

$$W_- := C \times I - \text{Int im } v_3 \quad \text{and} \quad W := W_- \cup_{v_3|_{S^5 \times S^2}} D^6 \times S^2.$$

Consider the cohomology exact sequence of pair (W, W_-) in the following Poincaré dual form (analogous to the sequence $(*)$ in [CSI, Proof of Lemma 4.8]):

$$\begin{array}{ccccc} H_6(D^6 \times S^2) & \longrightarrow & H_6(W, \partial) & \xrightarrow{r_{W_-}} & H_6(W_-, \partial) \longrightarrow H_5(D^6 \times S^2) . \\ PD_{\text{dex}} \uparrow \cong & & & & PD_{\text{dex}} \uparrow \cong \\ H^2(W, W_-) & & & & H^3(W, W_-) \end{array}$$

Since $H_5(D^6 \times S^2) = 0$, the map r_{W_-} is an epimorphism. Take any

$$Z \in r_{W_-}^{-1}(A[N] \times I \cap W_-) \subset H_6(W, \partial).$$

Denote

$$\widehat{V} := V \cup D^6 \times 1_2 \quad \text{and} \quad z := Z + [\widehat{V}] \in H_6(W, \partial).$$

Since $H_5(D^6 \times S^2) = 0$, the spin structure on W_- coming from $S^7 \times I$ extends to W . Clearly, $\partial W \underset{\text{spin}}{=} \partial(C \times I) \underset{\text{spin}}{=} M$ (for the ‘boundary’ spin structure on M coming from $C \times I$). Since

$$\partial_W Z = \partial_{C \times I}(A[N] \times I) = Y_0 \quad \text{and} \quad \partial_W [\widehat{V}] = [\nu^{-1}P \times \frac{1}{2}] = i_M \widehat{A}y, \quad \text{we have} \quad \partial_W z = Y_y.$$

By [CSI, Lemma 4.7] $\partial_W z^2 = Y_y^2 = 0$. So $z^2 \in \text{im } j_W$. Analogously to $(*)$ we obtain an isomorphism $H_4(C \times I) \cong H_4(W)$ commuting with $i_{C \times I} : H_4(M) \rightarrow H_4(C \times I)$ and $i_W : H_4(M) \rightarrow H_4(W)$. Since $i_{C \times I}$ is onto, i_W is onto. Hence $j_W = 0$. Thus $z^2 = 0$. So take $\overline{z^2} := 0$ and obtain $\eta(\tau_0, y) = \overline{z^2} \cap_W (z^2 - p_W^*) = 0$. (We essentially proved that if $\widehat{A}_f y$ is spherical, then $\eta(f, y) = 0$.) \square

3.2 Proof of Lemma 2.1.b

In this and the following subsection $f, f_0, f_1 : N \rightarrow S^7$ are embeddings representing any elements of $(\varkappa \times \lambda)^{-1}(u, l)$; we denote $d := \text{div}(u)$.

Definition of W', W'_- and $i' : W' \rightarrow W$. Let

$$W'_- := C_f - \text{Int im } v_2, \quad W' := W'_- \cup_{v_2|_{S^2 \times S^3 \times S^1}} S^2 \times D^4 \times S^1,$$

(Manifold W' may be called the result of an S^1 -parametric surgery along v_2 .) Define an embedding $W'_- \rightarrow W_-$ by $x \mapsto x \times 1/2$. We assume that this embedding and the standard embedding $S^2 \times D^4 \times S^1 \rightarrow S^2 \times D^4 \times S^2$ (that is the product of the identity and the equatorial inclusion $S^1 \rightarrow S^2$) fit together to give an embedding

$$i' : W' \rightarrow W.$$

Observe that $\Delta, \widehat{V} \subset W'$.

Lemma 3.2. *For each $y \in H_3$ and W_-, z, Z, V defined in [CSI, Proof of Lemma 4.8] we have*

$$z^2 \cap_W W_- \equiv_d 2i_{V, W_-}(Z \cap V) \in H_4(W_-, \partial)$$

(since $\partial V \subset \partial W_-$, the inclusion induces a map $i_{V, W_-} : H_4(V, \partial) \rightarrow H_4(W_-, \partial)$).

Proof. Since $\widehat{V} \subset W'$, we have $[\widehat{V}]^2 = 0 \in H_4(W, \partial)$. Also

$$Z^2 \cap W_- = (A_f[N] \times I)^2 \cap W_- = A_f \varkappa(f) \times I \cap W_- \equiv_d 0 \in H_4(W_-, \partial).$$

Hence

$$z^2 \cap W_- = (Z + [\widehat{V}])^2 \cap W_- \equiv_d 2(Z \cap_W [\widehat{V}]) \cap W_- = 2i_{V, W_-}(Z \cap \widehat{V} \cap W_-) = 2i_{V, W_-}(Z \cap V).$$

□

Proof of Lemma 2.1.b. In this proof a statement or a construction involving k holds or is made for each $k = 0, 1$. Given $y_k \in \ker(2\rho_d \bar{l})$ construct manifold W_k as W of [CSI, Proof of Lemma 4.8] by parametric surgery in $C_f \times [k-1, k]$. We add subscript k to $W_-, W'_-, t, \Delta, Z, \widehat{V}, z$ constructed in [CSI, Proof of Lemma 4.8]. (So unlike in other parts of this paper, subscript 0 of a manifold does not mean deletion of a codimension 0 ball from the manifold.) Define

$$W := W_0 \cup_{C_f \times 0} W_1 \quad \text{and} \quad W_- := C_f \times [-1, 1] - \text{Int im}(v_{3,0} \sqcup v_{3,1}) = W_{0-} \cup_{C_f \times 0} W_{1-}.$$

This W should not be confused with what were previously denoted W but now is denoted W_0 and W_1 . Same remark should be done for W_- and for Z, V, \widehat{V}, z constructed below.

The spin structure on W_- coming from $S^7 \times [-1, 1]$ extends to W . Clearly, $\partial W \stackrel{\text{spin}}{=} \partial(C_f \times [-1, 1]) \cong_{\text{spin}} M_f$ (for the ‘boundary’ spin structure on $\partial(C_f \times [-1, 1])$ and on M_f).

Since $H_5(t_k \times \Delta_k) = 0$, by the cohomological exact sequence of the pair (W, W_-) (cf. diagram $(*)$ in [CSI, Proof of Lemma 4.8]), $r_{W_-} : H_6(W, \partial) \rightarrow H_6(W_-, \partial)$ is an epimorphism. Take any

$$Z \in r_{W_-}^{-1}(A_f[N] \times [-1, 1] \cap W_-) \subset H_6(W, \partial).$$

Denote

$$V := V_0 \sqcup V_1, \quad \widehat{V} := \widehat{V}_0 \sqcup \widehat{V}_1 \quad \text{and} \quad z := Z + [\widehat{V}] \in H_6(W, \partial).$$

Since $\partial_W Z = Y_{f,0}$ and $\partial_W [\widehat{V}_k] = i_{\partial W} \widehat{A}_f y_k$, we have $\partial z = Y_{f,y_0+y_1}$. Thus the pair (W, z) is a spin null-bordism of (M_f, Y_{f,y_0+y_1}) .

Since $y_k \in \ker(2\rho_d \bar{l})$, we have $\partial z_k^2 \equiv_d 0$. Take any $\overline{z_k^2} \in j_{W_k}^{-1} \rho_d z_k^2$. Let

$$\overline{z^2} := i_{W_0, W} \overline{z_0^2} + i_{W_1, W} \overline{z_1^2}.$$

Then $\overline{z^2} \cap W_k = \overline{z_k^2}$. Also

$$\begin{aligned} j_W \overline{z^2} \cap W_- &= \sum_{k=0}^1 j_{W_k} \overline{z_k^2} \cap W_- = \rho_d \sum_{k=0}^1 z_k^2 \cap W_{k-} \quad \text{and} \\ \sum_{k=0}^1 z_k^2 \cap W_{k-} &\stackrel{(1)}{\equiv} 2 \sum_{k=0}^1 i_{V_k, W_{k-}} (Z_k \cap V_k) = 2i_{V, W_-} (Z \cap V) \stackrel{(3)}{\equiv} z^2 \cap W_-. \end{aligned}$$

Here congruences (1) and (3) modulo d hold by Lemma 3.2 and analogously to Lemma 3.2, respectively.

Hence by the cohomological exact sequence of the pair (W, W_-) with coefficients \mathbb{Z}_d (cf. diagram (*) in [CSI, Proof of Lemma 4.8]) $j_W \overline{z^2} - \rho_d z^2 = n_0[t_0] + n_1[t_1]$ for some $n_0, n_1 \in \mathbb{Z}_d$. We have

$$n_k[t_k] = (j_W \overline{z^2} - \rho_d z^2) \cap W_k = j_{W_k} \overline{z_k^2} - \rho_d z_k^2 = 0 \in H_4(W_k, \partial; \mathbb{Z}_d).$$

Therefore $n_0 = n_1 = 0$. So $j_W \overline{z^2} = \rho_d z^2$.

Since $\widetilde{W}_k := W_k - C_f \times [0, (2k-1)/3]$ is a deformation retract of W_k , the inclusion $\widetilde{W}_k \rightarrow W_k$ induces an isomorphism on H_4 . Clearly, $z \cap W_k = z_k$, so $z^2 \cap W_k = z_k^2$. Hence

$$\overline{z^2} \cap_W (z^2 - p_W) = \sum_{k=0}^1 (\overline{z^2} \cap W_k) \cap_{W_k} ((z^2 - p_W) \cap W_k) = \sum_{k=0}^1 \overline{z_k^2} \cap_{W_k} (z_k^2 - p_{W_k}).$$

So $\eta(f, \cdot)$ is a homomorphism. □

3.3 Proof of Lemma 2.1.ac

Lemma 3.3. *For each $y \in H_3$ and W, Z, V, t, Δ defined in [CSI, Proof of Lemma 4.8] we have*

(a) $\partial(Z \cap V) = [\partial\Delta] - i_{\partial C_f, \partial V} \xi y \in H_3(\partial V)$, where $\xi : N_0 \rightarrow \partial C_f$ is a weakly unlinked section for f (see definition in [CSI, §2.2]).

(b) $p_W = 2m[t] \in H_4(W, \partial)$ for some $m \in \mathbb{Z}$.

Lemma 3.3.b is essentially proved in the proof of [CSI, Lemma 4.8].

Proof of (a). The equality follows because

$$Z \cap V = (Z \cap W_-) \cap V = (A_f[N] \times I) \cap V = A_f[N] \cap V \in H_4(V, \partial) \quad \text{and}$$

$$\partial(A_f[N] \cap V) = A_f[N] \cap \partial V = i_{\partial V} (A_f[N] \cap \text{im } v) - i_{\partial V} (A_f[N] \cap \nu_f^{-1} P) \stackrel{(3)}{\equiv} [\partial\Delta] - [\xi P].$$

Here P and v are defined in [CSI, Proof of Lemma 4.8]. Equality (3) follows because

- $A_f[N] \cap \nu_f^{-1}P = [\xi P]$ by [CSI, Lemma 3.2.a].
- $A_f[N] \cap \text{im } v = [v(1_2 \times S^3)] = [\partial\Delta]$ since

$$(A_f[N] \cap \text{im } v) \cap_{\text{im } v} [v(S^2 \times 1_3)] = A_f[N] \cap_{C_f} v(S^2 \times 1_3) \stackrel{(2)}{=} A_f[N] \cap_{C_f} S_f^2 = 1.$$

Here equality (2) holds because $v(S^2 \times 1_3)$ is homologous to S_f^2 in C_f . \square

Lemma 3.4. *For each $y \in \ker(2\rho_d\bar{l})$ and W, W_-, z, t defined in [CSI, Proof of Lemma 4.8] there is a class $\widehat{z^2} \in H_4(W; \mathbb{Z}_d)$ such that*

- (a) $\overline{z^2} := \widehat{z^2} + n[t] \in j_W^{-1}\rho_d z^2 \subset H_4(W, \partial; \mathbb{Z}_d)$ for some $n \in \mathbb{Z}_d$.
- (b) $[t]^2 = (\widehat{z^2})^2 = 0 \in \mathbb{Z}_d$ and $[t] \cap_W \widehat{z^2} = 2 \in \mathbb{Z}_d$.

The proof is given later in this section. ⁵

Proof of Lemma 2.1.a. The Lemma follows by [CSI, Lemma 4.8] and Lemmas 3.3.b, 3.4. Indeed,

$$\overline{z^2} \cap_W (z^2 - p_W^*) = \overline{z^2} \cap_W \overline{z^2} - \overline{z^2} \cap_W p_W^* \stackrel{(2)}{=} (\widehat{z^2} + n[t])^2 - (\widehat{z^2} + n[t]) \cap_W 2m[t] \stackrel{(3)}{=} 4n - 4m.$$

Here

- equality (2) holds by Lemma 3.3.b and property (a) of Lemma 3.4,
- equality (3) holds by property (b) of Lemma 3.4. \square

In the proof of Lemma 2.1.c we will use not only the statement of Lemma 3.4 but also the following definition, which is also used in the proof of Lemma 3.4.

Definition of $a, s, \widehat{z^2}$ for $y \in \ker(2\rho_d\bar{l})$. By Lemma 3.3.a there is a representative

$$a \in C_4(V) \quad \text{of} \quad Z \cap V \in H_4(V, \partial) \quad \text{such that} \quad \partial a = \partial\Delta - \xi P.$$

(Such a representative is obtained from a representative $a' \in C_4(V)$ of $Z \cap V \in H_4(V, \partial)$ such that $\partial a' = \partial\Delta - \xi P + \partial a''$ for some $a'' \in C_4(\partial V)$ by the formula $a := a' - a''$.)

Since $y \in \ker(2\rho_d\bar{l})$, by [CSI, Lemma 3.2.λ] there is a chain

$$s \in C_4(C_f \times 0; \mathbb{Z}_d) \quad \text{such that} \quad \partial s = 2\xi P \times 0.$$

Define

$$\widehat{z^2} := [2a - 2\Delta - 2\xi P \times [0, \frac{1}{2}] + s] \in H_4(W; \mathbb{Z}_d).$$

Proof of Lemma 3.4. We have

$$\rho_d z^2 \cap W_- \stackrel{(1)}{=} 2\rho_d i_{V, W_-}(Z \cap V) = [2a]_{W_-, \partial} = [2a - 2\xi P \times [0, \frac{1}{2}] + s]_{W_-, \partial} = \widehat{z^2} \cap W_- = j_W \widehat{z^2} \cap W_-,$$

where equality (1) follows by Lemma 3.2. Hence by the cohomology exact sequence of pair (W, W_-) (cf. diagram (*) in [CSI, Proof of Lemma 4.8]) $\rho_d z^2 = j_W(\widehat{z^2} + n[t])$ for some $n \in \mathbb{Z}_d$. Thus property (a) holds.

⁵Equality (3) from [CSI, Proof of Lemma 4.3.a] also holds by [CSI, Proof of Lemma 4.7] and 3.4 because by Lemma 3.4.a for $d = 2$ we have $\eta'(\text{id } \partial C_{f_0}, Y_{f_0, y}) = \overline{z^2} \cap_W z^2 = \overline{z^2} \cap_W \overline{z^2} = 2[t] \cap_W \widehat{z^2} = 0 \in \mathbb{Z}_2$.

Let us prove property (b). We have $[t]^2 = [S^2 \times 0 \times S^2] \cap_{S^2 \times D^3 \times S^2} [S^2 \times 1_3 \times S^2] = 0$. Since the support of \widehat{z}^2 is in $W' \cup \partial C_f \times [0, \frac{1}{2}] \cup C_f \times 0$ and this space is the boundary of a connected component of $W - W'$, we have $(\widehat{z}^2)^2 = 0$. Also

$$[t] \cap_W \widehat{z}^2 = [t] \cap_{W_-} (\widehat{z}^2 \cap W_-) = [t] \cap_{W_-} [2a]_{W_-, \partial} = 2[t] \cap_{\partial W_-} [\partial a] = 2[t] \cap_{t \times \partial \Delta} [\partial \Delta] = 2.$$

Here the homology classes are taken in the space indicated under ' \cap ' (so $[t]$ has different meaning in different parts of the formula), and $\widehat{z}^2 \cap W_- = [2a]_{W_-, \partial}$ is proved in the proof of (a). \square

Proof of Lemma 2.1.c. Take any bundle isomorphism $\varphi : \partial C_0 \rightarrow \partial C_1$ given by [CSI, Lemma 2.5]. Take a closed oriented 3-submanifold $P \subset N$ realizing $y \in H_{f_0} = H_{f_1}$. For $k = 0, 1$ construct maps v_{jk} , $j = 0, 1, 2, 3$, manifolds $V_k \subset C_k$, \widehat{V}_k , W'_k and W_k , chains a_k, s_k and classes $Z_k, z_k, \widehat{z}_k^2$ as in [CSI, Proof of Lemma 4.8] and above. (So unlike in other parts of this paper, subscript 0 of a manifold does not mean deletion of a codimension 0 ball from the manifold.) Define

$$W := W_0 \cup_{\varphi \times \text{id} : \partial C_0 \times I \rightarrow \partial C_1 \times I} W_1.$$

This W should not be confused with what was previously denoted W but now is denoted W_0 and W_1 . Same remark should be done for z, Z, \widehat{V} constructed below.

Consider the following segment of the ('cohomological') Mayer-Vietoris sequence:

$$H_6(W, \partial) \xrightarrow{r_{W_0} \oplus r_{W_1}} H_6(W_0, \partial) \oplus H_6(W_1, \partial) \xrightarrow{r_0 \oplus (-r_1)} H_4(\partial C_0).$$

Here r_k is the composition $H_6(W_k, \partial) \xrightarrow{\partial} H_5(\partial W_k) \xrightarrow{r_{\partial C_0}} H_4(\partial C_0)$. We have

$$r_k Z_k = (\partial Z_k) \cap \partial C_0 = Y_{f_k} \cap \partial C_0 = \partial(Y_{f_k} \cap C_k) \stackrel{(4)}{=} \partial A_k[N] \stackrel{(5)}{=} \partial A_{1-k}[N] \stackrel{(6)}{=} r_{1-k} Z_{1-k} \in H_4(\partial C_0).$$

Here

- equality (4) holds by descriptions of joint Seifert classes [CSI, Lemma 3.13.a].
- equality (5) holds by agreement of Seifert classes [CSI, Lemma 3.5.a]
- equality (6) holds analogously to the previous set of equalities.

Hence there exists $Z \in H_6(W, \partial)$ such that $Z \cap W_k = Z_k$. Denote

$$\widehat{V} := \widehat{V}_0 \bigcup_{\varphi : \nu_0^{-1} P \rightarrow \nu_1^{-1} P} \widehat{V}_1 \subset W' \quad \text{and} \quad z := Z + [\widehat{V}] \in H_6(W, \partial).$$

Clearly, $z \cap W_k = z_k$.⁶

Take $\widehat{z}_k^2 \in H_4(W; \mathbb{Z}_d)$ given by Lemma 3.4. Then by Lemmas 3.3.b and 3.4

$$\overline{z}_k^2 \cap_W p_W^* = 4m_k = \widehat{z}_k^2 \cap_W p_W^* \quad \text{and} \quad \overline{z}_k^2 \cap_W z_k^2 = \overline{z}_k^2 \cap_W \overline{z}_k^2 = 4n_k = 2\widehat{z}_k^2 \cap_W \overline{z}_k^2 = 2\widehat{z}_k^2 \cap_W z_k^2.$$

Hence

$$\eta(f_k, y) = \rho_d(\widehat{z}_k^2 \cap_{W_k} (2z_k^2 - p_{W_k}^*)) = \rho_d(\widehat{z}_k^2 \cap_W (2z^2 - p_W^*)).$$

Take a weakly unlinked section $\xi_0 : N_0 \rightarrow \partial C_0$ of f_0 . By [CSI, Lemma 3.4] $\xi_1 := \varphi \xi_0$ is an unlinked section of f_1 . Hence

$$\partial a_1 - \partial \Delta_1 = -\xi_1 P = -\xi_0 P = \partial a_0 - \partial \Delta_0 \quad \text{and} \quad \partial s_1 = 2\xi_1 P = 2\xi_0 P = \partial s_0.$$

⁶Note that (W, z) is not assumed to be spin bordism of anything and possibly $\rho_d \partial z^2 \neq 0$.

Identify M_φ and its subsets with $M_\varphi \times 0 \subset \partial W$ and with corresponding subsets. Denote

$$\widehat{a} := [\Delta_0 - a_0 + a_1 - \Delta_1] \in H_4(\widehat{V}; \mathbb{Z}_d) \quad \text{and} \quad s := [s_0 - s_1] \in H_4(M_\varphi; \mathbb{Z}_d).$$

Then by definition of \widehat{z}_k^2

$$\widehat{z}_0^2 - \widehat{z}_1^2 = i_\varphi s - 2i\widehat{a}, \quad \text{where} \quad i_\varphi := i_{M_\varphi, W} \quad \text{and} \quad i := i_{\widehat{V}, W}.$$

We have $i_\varphi s \cap_W p_W^* = s \cap_{M_\varphi} p_{M_\varphi}^* = 0$.

Since

$$(z \cap M_\varphi) \cap_{M_\varphi} S_{f_0}^2 = (\partial z_0 \cap C_0) \cap_{C_0} S_{f_0}^2 = Y_{f_0} \cap_{C_0} S_{f_0}^2 = 1,$$

$z \cap M_\varphi$ is a joint Seifert class for φ . Then

$$i_\varphi s \cap_W z^2 = (s \cap \partial C_0) \cap_{\partial C_0} (z^2 \cap \partial C_0) \stackrel{(2)}{=} 2\xi_0 y \cap_{\partial C_0} \nu_0^! \beta = 2\beta \cap_N y,$$

where

- $\beta \in H_1(N; \mathbb{Z}_d)$ is a lifting of $\beta(f_0, f_1)$;
- equality (2) follows because $s \cap \partial C_0 = 2[\xi_0 P] = 2\xi_0 y$ and because $z^2 \cap \partial C_0 = (z \cap M_\varphi)^2 \cap \partial C_0 = \nu_0^! \beta$ by definition of $\beta(f_0, f_1)$.

We have

$$\begin{aligned} z^2 \cap_W i\widehat{a} &\stackrel{(1)}{=} (Z + [\widehat{V}])^2 \cap_W i\widehat{a} \stackrel{(2)}{=} Z^2 \cap_W i\widehat{a} + 2Z \cap_W [\widehat{V}] \cap_W i\widehat{a} \stackrel{(3)}{=} \\ &= (Z \cap \widehat{V})_{\widehat{V}}^2 \cap_{\widehat{V}} \widehat{a} + 2i(Z \cap \widehat{V}) \cap_W i\widehat{a} \stackrel{(4)}{=} (\widehat{a})_{\widehat{V}}^3 + 2(i\widehat{a})^2 \stackrel{(5)}{=} (\widehat{a})_{\widehat{V}}^3, \end{aligned}$$

where

- equality (1) follows by definition of z ;
- equalities (2) and (5) follow because $\widehat{V} \subset W'$, so $[\widehat{V}]^2 = 0$ and $(i\widehat{a})^2 = 0$;
- equality (3) is obvious;
- equality (4) follows because $Z \cap \widehat{V} = \widehat{a}$ by definition of a_0, a_1, \widehat{a} .

Therefore $i\widehat{a} \cap_W (2z^2 - p_W^*) = 2(\widehat{a})_{\widehat{V}}^3 - \widehat{a} \cap_{\widehat{V}} p_{\widehat{V}}^* \stackrel{12}{=} 0$ by [Wa66, Theorem 5].

Now the Lemma follows because

$$(\widehat{z}_0^2 - \widehat{z}_1^2) \cap_W (2z^2 - p_W^*) = 2i_\varphi s \cap_W z^2 - i_\varphi s \cap_W p_W^* - 2i\widehat{a} \cap_W (2z^2 - p_W^*) \stackrel{24}{=} 4\beta \cap_N y.$$

□

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